CONNES AMENABILITY OF THE SECOND DUAL OF ARENS REGULAR BANACH ALGEBRAS

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ABSTRACT. In this paper we study the Connes amenability of the second dual of Arens regular Banach algebras. Of course we provide a partial answer to the question posed by Volker Runde. Also we fined the necessary and sufficient conditions for the second dual of an Arens regular module extension Banach algebra to be Connes amenable when the module is reflexive.

Introduction

A Banach algebra $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = {\mathcal{A}_*}^*$. Let $\mathcal{A}$ be a dual Banach algebra. A dual Banach $\mathcal{A}$-module X is called normal if, for every $x \in X$, the maps $a \longmapsto a.x$ and $a \longmapsto x.a$ are $weak^* - weak^*$-continuous from $\mathcal{A}$ into X.

For example if G is a locally compact topological group, then $M(G)$ is a dual Banach algebra with predual $C_0(G)$. Also if $\mathcal{A}$ is an Arens regular Banach algebra, then $\mathcal{A}^{**}$ (by the first Arens product) is a dual Banach algebra with predual $\mathcal{A}^*$. Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-module then a derivation from $\mathcal{A}$ into X is a linear map D, such that for every $a, b \in \mathcal{A}$, $D(ab) = D(a).b + a.D(b)$. Let $x \in X$, and let $\delta_x : \mathcal{A} \longrightarrow X$ defined by $\delta_x(a) = a.x - x.a \quad (a \in \mathcal{A})$, then δ_x is a derivation, derivations of this form are called inner derivations. A Banach algebra is called amenable if every derivation from $\mathcal{A}$ into each dual $\mathcal{A}$-module is inner; i.e. $H^1(\mathcal{A}, X^*) = \{o\}$, foe every $\mathcal{A}$-module X. This definition was introduced by B. E. Johnson in [4]. A dual Banach algebra $\mathcal{A}$ is Connes amenable if every $weak^* - weak^*$-continuous derivation from $\mathcal{A}$ into each normal dual Banach $\mathcal{A}$-module X is inner; i.e. $H^1_{w^*}(\mathcal{A}, X) = \{o\}$, this definition was introduced by V.Runde (see section 4 of [6]). We answer partially to the following question [6, 2].

Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}^{**}$ is Connes amenable. Need $\mathcal{A}$ be amenable?

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1. Second dual of Arens regular Banach algebras

Let the Banach algebra $\mathcal{A}$ be Arens regular, then we have the following assertions.
(i) If $\mathcal{A}$ is amenable, then $\mathcal{A}^{**}$ is Connes amenable.
(ii) If $\mathcal{A}$ is an ideal of $\mathcal{A}^{**}$ and $\mathcal{A}^{**}$ is Connes amenable, then $\mathcal{A}$ is amenable [5].
First we have the following result.

Theorem 1.1. Let $\mathcal{A}$ be a Banach algebra which A^{**} is Arens regular and $\mathcal{A}^{****}$ is Connes amenable. Then $\mathcal{A}^{**}$ is Connes amenable. Also if $\mathcal{A}^{**}$ is an ideal of $\mathcal{A}^{****}$, then $\mathcal{A}^{**}$ is amenable and reflexive.

Proof. Let X be a normal $\mathcal{A}^{**}$-module, and let $\pi : a'''' \longmapsto a'''' \mid_{\mathcal{A}^*}: \mathcal{A}^{****} \longrightarrow \mathcal{A}^{**}$ be the restriction map. Since π is $weak^* - weak^*$-continuous, then X is a normal $\mathcal{A}^{****}$-module by the following module actions

$$a''''x = \pi(a'''')x, \qquad xa'''' = x\pi(a'''') \qquad (x \in X, a'''' \in \mathcal{A}^{****}).$$

Let $D : \mathcal{A}^{**} \longrightarrow X$ be a $weak^* - weak^*$-continuous derivation. It is easy to show that $Do\pi : \mathcal{A}^{****} \longrightarrow X$ is a $weak^* - weak^*$-continuous derivation. If $\mathcal{A}^{****}$ is Connes amenable then $Do\pi$ is inner, so D is inner, and $\mathcal{A}^{**}$ is Connes amenable. Connes amenability of $\mathcal{A}^{**}$ implies that $\mathcal{A}^{**}$ is unital. Let now $\mathcal{A}^{**}$ be an ideal of $\mathcal{A}^{****}$, then $\mathcal{A}^{**} = \mathcal{A}^{****}$. Thus $\mathcal{A}^{**}$ is reflexive, also by theorem 4.4 of [5], $\mathcal{A}^{**}$ is amenable. ■

Corollary 1.2. Let $\mathcal{A}$ be a Banach algebra which A^{**} is amenable and Arens regular. If $\mathcal{A}^{**}$ is an ideal of $\mathcal{A}^{****}$, then $\mathcal{A}$ is reflexive.

Theorem 1.3. Let $\mathcal{A}$ be an Arens regular Banach algebra with a bounded approximate identity, which is a right ideal of A^{**}. Let for every $\mathcal{A}^{**}$-neo unital module X, X^* factors $\mathcal{A}$ on the left, i.e. $\mathcal{A}X^* = X^*$. If $\mathcal{A}^{**}$ is Connes amenable, then $\mathcal{A}$ is amenable.

Proof. Let X be a $\mathcal{A}$-module, and let $D : \mathcal{A} \longrightarrow X^{**}$ be a derivation. Since $\mathcal{A}^{**}$ is Connes amenable, then $\mathcal{A}^{**}$ has unite element E. We can extend the actions of $\mathcal{A}$ on X^{**} to actions of $\mathcal{A}^{**}$ on X^{****} via

$$a''.x'''' = w^*\text{-}\lim_i \lim_j a_i\, x''_j$$

and

$$x''''.a'' = w^*\text{-}\lim_j \lim_i x''_j\, a_i,$$

where $a'' = w^*\text{-}\lim_i a_i$, $x'''' = w^*\text{-}\lim_j x''_j$. We have the direct sum decomposition

$$X^{****} = EX^{****}E \oplus (1-E)X^{****}E \oplus EX^{****}(1-E) \oplus (1-E)X^{****}(1-E),$$

as $\mathcal{A}^{**}$-modules. For i=1,2,3,4, let π_i be the associated projection and let $D_i = \pi_i o D^{**}$. π_i is a $\mathcal{A}^{**}$-module morphism, then D_i is a derivation. Let $y_2 = -D_2(E)$. Since $D^{**}(a'') = (D^{**}(E))a'' - ED^{**}(a'')$ and $a''D_2(E) = a''(1-E)(D^{**}(E))E = 0$, then

$$D_2(a'') = (1-E)(D^{**}(E))a''E = D_2(E)a'' = \delta_{y_2}(a'').$$

A similar argument applies to D_3 and D_4. We show that $EX^{****}E$ is a normal $\mathcal{A}^{**}$-module. First we have

$$EX^{****}E = (EX^{***}E)^{*} \qquad (1).$$

Let $Ex''''E \in EX^{****}E$ and let $a''_\alpha \xrightarrow{weak^*} a''$ in $\mathcal{A}^{**}$. Since $EX^{***}E$ is neo-unital $\mathcal{A}^{**}$- module, then $EX^{****}E = EX^{****}E\mathcal{A}$, therefore there is $a \in \mathcal{A}$ and $y'''' \in X^{****}$ such that $Ey''''Ea = Ex''''E$. We have $aa''_\alpha \xrightarrow{weak^*} aa''$ in $\mathcal{A}^{**}$, since $\mathcal{A}$ is a right ideal of A^{**}, then $aa''_\alpha \xrightarrow{weak} aa''$ in $\mathcal{A}$. Thus by (1), we have

$$Ex''''Ea''_\alpha = Ey''''Eaa''_\alpha \xrightarrow{weak} Ey''''Eaa'' = Ex''''Ea'' \qquad in \quad EX^{****}E.$$

Then

$$Ex''''Ea''_\alpha \xrightarrow{weak^*} Ex''''Ea'' \qquad in \quad EX^{****}E.$$

Trivially we have

$$a''_\alpha Ex''''E \xrightarrow{weak^*} a''Ex''''E \qquad in \quad EX^{****}E.$$

This means that $EX^{****}E$ is a normal $\mathcal{A}^{**}$-module. Since π_1 and D^{**} are $weak^* - weak^*$-continuous, and $\mathcal{A}^{**}$ is Connes amenable, then there is $x_1'''' \in X^{****}$ such that $D_1 = \delta_{Ex_1''''E}$. Thus D^{**} is inner. On the other hand we have the following direct sum decomposition of $\mathcal{A}$-moduls

$$X^{****} = \widehat{X^{**}} \oplus \widehat{(X^{*})}^{\perp}.$$

Let $\pi : X^{****} \longrightarrow X^{**}$ be the natural projection, then $D = \pi o D^{**}$ is inner. Thus $H^1(\mathcal{A}, X^{**}) = \{o\}$, and by Proposition 2.8.59 of [1], $\mathcal{A}$ is amenable. ■

2. Module extension dual Banach algebras

Let $\mathcal{A}$ be a Banach algebra and M be a Banach $\mathcal{A}$-module (with module actions π_r and π_l), let $\mathcal{B} = M \oplus_1 \mathcal{A}$ as a Banach space, so that

$$\|(m,a)\| = \|m\| + \|a\| \qquad (a \in \mathcal{A}\ ,\ m \in M)\ .$$

Then $\mathcal{B}$ is a Banach algebra with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 \cdot a_2 + a_1 \cdot m_2, a_1 a_2)\ .$$

The second dual $\mathcal{B}^{**}$ of $\mathcal{B}$ is identified with $M^{**} \oplus_1 \mathcal{A}^{**}$ as a Banach space and the first Arens product on $\mathcal{B}^{**}$ is given by

$$(m_1'', a_1'')(m_2'', a_2'') = (m_1'' \cdot a_2'' + a_1'' \cdot m_2'', a_1'' a_2'') \ .$$

As in [3] we can show that B is Arens regular if and only if for every $a'' \in \mathcal{A}^{**}$, and $m'' \in M^{**}$,
(1) $b'' \longmapsto a''b'' : \mathcal{A}^{**} \longrightarrow \mathcal{A}^{**}$ is $weak^* - weak^*$ continuous.
(2) $n'' \longmapsto a''n'' : M^{**} \longrightarrow M^{**}$ is $weak^* - weak^*$ continuous.
(3) $b'' \longmapsto m''b'' : \mathcal{A}^{**} \longrightarrow M^{**}$ is $weak^* - weak^*$ continuous.
Then we have the following theorem.

Theorem 2.1. Let $\mathcal{A}$ be an Arens regular Banach algebra, and let M be a reflexive Banach $\mathcal{A}$-module. Then
(i) $\mathcal{B} = M \oplus_1 \mathcal{A}$ is Arens regular.
(ii) $\mathcal{B}^{**} = (M \oplus_1 \mathcal{A})^{**}$ is Connes amenable if and only if $M = 0$, and $\mathcal{A}^{**}$ is Connes amenable.

Proof. We can prove (i) by the argument above theorem. To prove (ii), suppose that $\mathcal{B}^{**}$ is Connes amenable, we need only to show that $M = 0$. Let $X = M^{***} \bigotimes_p \mathcal{A}^{**}$. We define the module actions of $\mathcal{B}$ on X as follows:

$$(m''' \otimes_p a'').(b'', x'') = m''' \otimes_p a''b'', \qquad (b'', x'').(m''' \otimes_p a'') = b''m''' \otimes_p a'',$$

so we define $D : \mathcal{B}^{**} \longrightarrow X^*$ by

$$\langle D((a'', x'')), m''' \otimes b'' \rangle = \langle x''m''', b'' \rangle.$$

Where $m''' \in M^{***}, x'' \in M^{**}$ and $a'', b'' \in \mathcal{A}^{**}$.

Let $(b''_\alpha, x''_\alpha) \xrightarrow{weak^*} (b'', x'')$ in $\mathcal{B}^{**}$, then we have $b''x''_\alpha \xrightarrow{weak^*} b''x''$ in M^{**}. Since M is reflexive then $b''x''_\alpha \xrightarrow{weakly} b''x''$ in M^{**}. Then for every $m''' \in M^{***}$ we have $\langle x''_\alpha m''', b'' \rangle \longrightarrow \langle x''m''', b'' \rangle$. This means that D is $weak^* - weak^*$ continuous. Also for every $(a_1'', x_1''), (a_2'', x_2'') \in \mathcal{B}^{**}, m''' \in M^{***}$ and $x'' \in M^{**}$, we have

$$\begin{aligned}\langle D((a_1'', x_1'')(a_2''x_2'')), m''' \otimes b'' \rangle =& \langle D((a_1''a_2'', x_1''a_2'' + a_1''x_2'')), m''' \otimes b'' \rangle \\ =& \langle (a_1''x_2''m''' + x_1''a_2''m'''), b'' \rangle = \langle x_2''m''', b''a_1'' \rangle + \langle x_1''a_2''m''', b'' \rangle \\ =& D((a_1'', x_1'')), (a_2''x_2'').(m''' \otimes b'')\rangle + D((a_2''x_2'')), (m''' \otimes b'').(a_1'', x_1''))\rangle \\ =& D((a_1'', x_1'')).(a_2''x_2''), m''' \otimes b'' \rangle + (a_1'', x_1'').D((a_2''x_2'')), m''' \otimes b'' \rangle.\end{aligned}$$

Thus D is a derivation. Since $\mathcal{B}^{**}$ is Connes amenable, than it is unital. Let (E, x_1'') be the unit element of $\mathcal{B}^{**}$, then it is easy to show that $x_1'' = 0$ and that E is unit element of $\mathcal{A}^{**}$, so

$Em'' = m''E = m''$ and $Em''' = m'''E = m'''$, for every $m'' \in M^{**}$ and $m''' \in M^{***}$. Since M is reflexive then it is easy to show that $\mathcal{A}M = M\mathcal{A} = M$ and the module actions of $\mathcal{B}^{**}$ on X^* are $weak^* - weak^*$ continuous. Then D is inner, and there exists $F \in X^*$ such that $D(a'', m'') = (a'', m'').F - F.(a'', m'')$. Then for every $m''' \otimes b'' \in X$, we have

$$\begin{aligned}
\langle m''m''', b''\rangle =& \langle\langle D((a'', m'')), m''' \otimes b''\rangle \\
=& \langle F, (m''' \otimes b'').(a'', m'')\rangle + \langle F, (a'', m'').(m''' \otimes b'')\rangle \\
=& \langle F, m''' \otimes b''a'' + a''m''' \otimes b''\rangle.
\end{aligned}$$

Let $a'' = 0$, then we have $m''m''' = 0$ for every $m''' \in M^{***}, m'' \in M^{**}$. This means that $\mathcal{A}^{**}M^{**} = 0$. Thus $m'' = Em'' = 0$ for every $m'' \in M^{**}$ and the proof is complete. ■

Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \Omega_{\mathcal{A}}$ be a multiplier on $\mathcal{A}$. Then $\mathbb{C}$ is a Banach $\mathcal{A}$-module by module actions

$$a.c = \varphi(a)c, \;\; c.a = c\varphi(a), \;\; (a \in \mathcal{A}, c \in \mathbb{C}).$$

We denote this $\mathcal{A}$-module with $\mathbb{C}_\varphi$. By apply above theorem we can give a class of dual Banach algebras which are not Conns amenable.

Corollary 2.2. Let $\mathcal{A}$ be an Arens regular Banach algebra, and let $0 \neq \varphi \in \Omega_{\mathcal{A}}$. Then $(\mathbb{C}_\varphi \oplus_1 \mathcal{A})^{**}$ is a dual Banach algebra which is not Connes amenable.

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